Solved Problems

6.1 Define *tilting* as a rotation about the x axis followed by a rotation about the y axis: (a) find the tilting matrix; (b) does the order of performing the rotation matter?

SOLUTION

(a) We can find the required transformation T by composing (concatenating) two rotation matrices:

$$T = R_{\theta_y,\mathbf{J}} \cdot R_{\theta_x,\mathbf{I}}$$

$$= \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_y & \sin\theta_y \sin\theta_x & \sin\theta_y \cos\theta_x & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ -\sin\theta_y & \cos\theta_y \sin\theta_x & \cos\theta_y \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) We multiply $R_{\theta_x,\mathbf{I}} \cdot R_{\theta_y,\mathbf{J}}$ to obtain the matrix

$$\begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ \sin\theta_x \sin\theta_y & \cos\theta_x & -\sin\theta_x \cos\theta_y & 0 \\ -\cos\theta_x \sin\theta_y & \sin\theta_x & \cos\theta_x \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is not the same matrix as in part a; thus the order of rotation matters.

6.2 Find a transformation $A_\mathbf{V}$ which aligns a given vector $\mathbf{V}$ with the vector $\mathbf{K}$ along the positive z axis.

SOLUTION

See Fig. 6-4(a). Let $\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. We perform the alignment through the following sequence of transformations [Figs. 6-4(b) and 6-4(c)]:

1. Rotate about the x axis by an angle θ_1 so that V rotates into the upper half of the xz plane (as the vector $\mathbf{V}_1$).
2. Rotate the vector $\mathbf{V}_1$ about the y axis by an angle $-\theta_2$ so that $\mathbf{V}_1$ rotates to the positive z axis (as the vector $\mathbf{V}_2$).

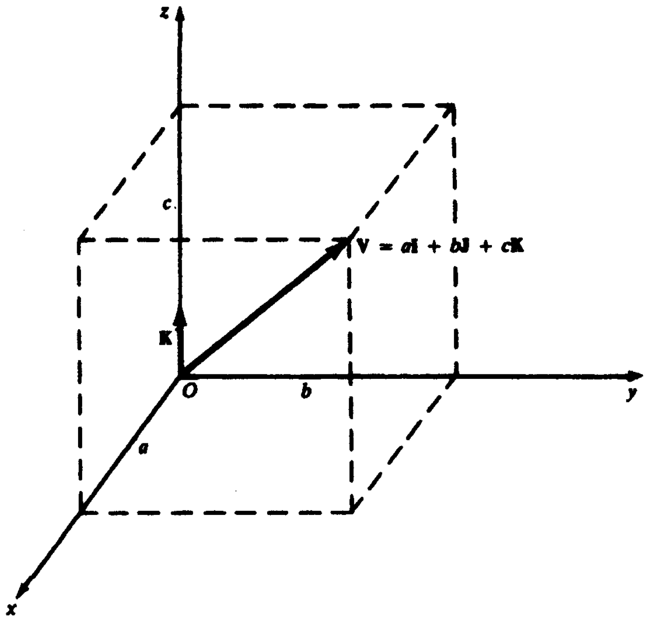
Implementing step 1 from Fig. 6-4(b), we observe that the required angle of rotation θ_1 can be found by looking at the projection of $\mathbf{V}$ onto the yz plane. (We assume that b and c are not both zero.) From triangle $OP'B$:

$$\sin\theta_1 = \frac{b}{\sqrt{b^2+c^2}} \qquad \cos\theta_1 = \frac{c}{\sqrt{b^2+c^2}}$$

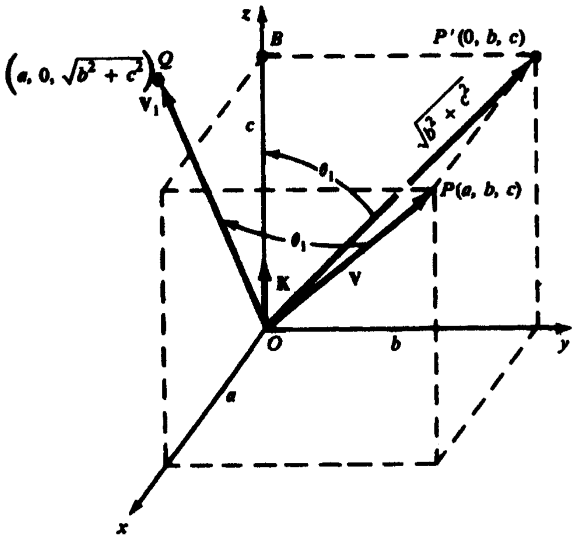
The required rotation is

$$R_{\theta_1,\mathbf{I}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{\sqrt{b^2+c^2}} & -\frac{b}{\sqrt{b^2+c^2}} & 0 \\ 0 & \frac{b}{\sqrt{b^2+c^2}} & \frac{c}{\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

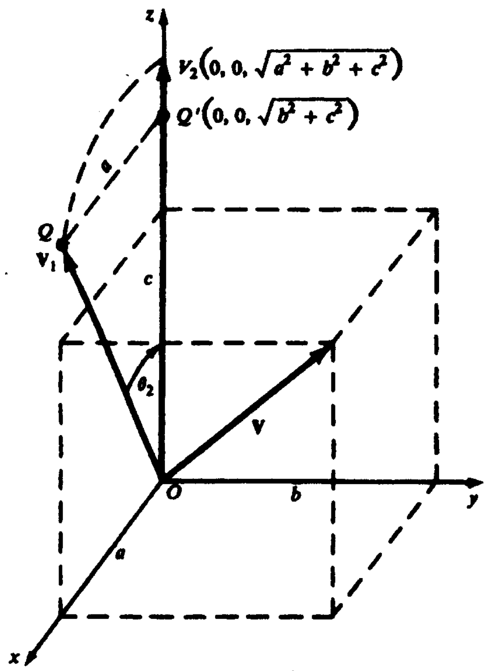
Applying this rotation to the vector $\mathbf{V}$ produces the vector $\mathbf{V}_1$ with the components $(a, 0, \sqrt{b^2+c^2})$.


z
x
y
O
a
b
c
K
V = aI + bJ + cK

(a)


z
x
y
O
a
b
c
B
K
V
V₁
Q
P'(0, b, c)
P(a, b, c)
θ₁

(b)


z
x
y
O
a
b
c
V
V₁
Q
θ₂
Q'(0, 0, √(b² + c²))
V₂(0, 0, √(a² + b² + c²))

(c)

Fig. 6-4

Implementing step 2 from Fig. 6-4(c), we see that a rotation of $-\theta_2$ degrees is required, and so from triangle OQQ':

$$\sin(-\theta_2) = -\sin\theta_2 = -\frac{a}{\sqrt{a^2+b^2+c^2}} \quad \text{and} \quad \cos(-\theta_2) = \cos\theta_2 = \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}$$

Then

$$R_{-\theta_2,\mathbf{J}} = \begin{pmatrix} \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 & \frac{-a}{\sqrt{a^2+b^2+c^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{\sqrt{a^2+b^2+c^2}} & 0 & \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $|\mathbf{V}| = \sqrt{a^2+b^2+c^2}$, and introducing the notation $\lambda = \sqrt{b^2+c^2}$, we find

$$A_{\mathbf{V}} = R_{-\theta_2,\mathbf{J}} \cdot R_{\theta_1,\mathbf{I}}$$

$$= \begin{pmatrix} \frac{\lambda}{|\mathbf{V}|} & \frac{-ab}{\lambda|\mathbf{V}|} & \frac{-ac}{\lambda|\mathbf{V}|} & 0 \\ 0 & \frac{c}{\lambda} & \frac{-b}{\lambda} & 0 \\ \frac{a}{|\mathbf{V}|} & \frac{b}{|\mathbf{V}|} & \frac{c}{|\mathbf{V}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If both b and c are zero, then $\mathbf{V} = a\mathbf{I}$, and so $\lambda = 0$. In this case, only a $\pm 90°$ rotation about the y axis is required. So if $\lambda = 0$, it follows that

$$A_{\mathbf{v}} = R_{-\theta_2,\mathbf{J}} = \begin{pmatrix} 0 & 0 & \frac{-a}{|a|} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{|a|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the same manner we calculate the inverse transformation that aligns the vector $\mathbf{K}$ with the vector $\mathbf{V}$:

$$A_{\mathbf{v}}^{-1} = (R_{-\theta_2,\mathbf{J}} \cdot R_{\theta_1,\mathbf{I}})^{-1} = R_{\theta_1,\mathbf{I}}^{-1} \cdot R_{-\theta_2,\mathbf{J}}^{-1} = R_{-\theta_1,\mathbf{I}} \cdot R_{\theta_2,\mathbf{J}}$$

$$= \begin{pmatrix} \frac{\lambda}{|\mathbf{V}|} & 0 & \frac{a}{|\mathbf{V}|} & 0 \\ \frac{-ab}{\lambda|\mathbf{V}|} & \frac{c}{\lambda} & \frac{b}{|\mathbf{V}|} & 0 \\ \frac{-ac}{\lambda|\mathbf{V}|} & -\frac{b}{\lambda} & \frac{c}{|\mathbf{V}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6.3 Let an axis of rotation L be specified by a direction vector $\mathbf{V}$ and a location point P. Find the transformation for a rotation of $\theta°$ about L. Refer to Fig. 6-5.

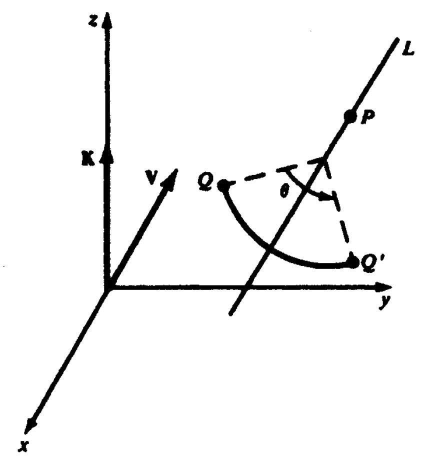


Fig. 6-5

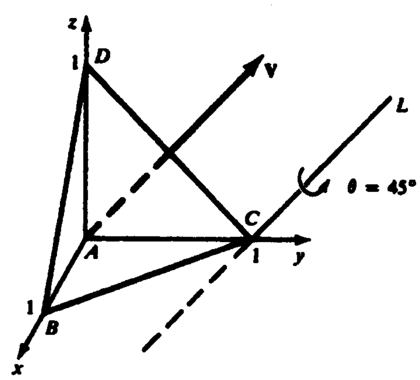


Fig. 6-6

SOLUTION

We can find the required transformation by the following steps:

1. Translate P to the origin.
2. Align $\mathbf{V}$ with the vector $\mathbf{K}$.
3. Rotate by $\theta°$ about $\mathbf{K}$.
4. Reverse steps 2 and 1.

So

$$R_{\theta,L} = T_{-P}^{-1} \cdot A_{\mathbf{V}}^{-1} \cdot R_{\theta,\mathbf{K}} \cdot A_{\mathbf{V}} \cdot T_{-P}$$

Here, $A_{\mathbf{v}}$ is the transformation described in Prob. 6.2.

6.4 The pyramid defined by the coordinates $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, and $D(0,0,1)$ is rotated 45° about the line L that has the direction $\mathbf{V} = \mathbf{J} + \mathbf{K}$ and passing through point $C(0,1,0)$ (Fig. 6-6). Find the coordinates of the rotated figure.

SOLUTION

From Prob. 6.3, the rotation matrix $R_{\theta,L}$ can be found by concatenating the matrices

$$R_{\theta,L} = T_{-P}^{-1} \cdot A_{\mathbf{V}}^{-1} \cdot R_{\theta,\mathbf{K}} \cdot A_{\mathbf{V}} \cdot T_{-P}$$

With $P = (0,1,0)$, then

$$T_{-P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now $\mathbf{V} = \mathbf{J} + \mathbf{K}$. So from Prob. 6.2, with $a = 0$, $b = 1$, $c = 1$, we find $\lambda = \sqrt{2}$, $|\mathbf{V}| = \sqrt{2}$, and

$$A_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Also

$$R_{45^\circ,\mathbf{K}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T_{-P}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$R_{\theta,L} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{2+\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} \\ -\frac{1}{2} & \frac{2-\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & \frac{\sqrt{2}-2}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To find the coordinates of the rotated figure, we apply the rotation matrix $R_{\theta,L}$ to the matrix of homogeneous coordinates of the vertices A, B, C, and D:

$$C = (ABCD) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

So

$$R_{\theta,L} \cdot C = \begin{pmatrix} \frac{1}{2} & \frac{1+\sqrt{2}}{2} & 0 & 1 \\ \frac{2-\sqrt{2}}{4} & \frac{4-\sqrt{2}}{4} & 1 & \frac{2-\sqrt{2}}{2} \\ \frac{\sqrt{2}-2}{4} & \frac{\sqrt{2}-4}{4} & 0 & \frac{\sqrt{2}}{2} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The rotated coordinates are (Fig. 6-7)

$$A' = \left(\frac{1}{2}, \frac{2-\sqrt{2}}{4}, \frac{\sqrt{2}-2}{4}\right) \qquad C' = (0, 1, 0)$$

$$B' = \left(\frac{1+\sqrt{2}}{2}, \frac{4-\sqrt{2}}{4}, \frac{\sqrt{2}-4}{4}\right) \qquad D' = \left(1, \frac{2-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

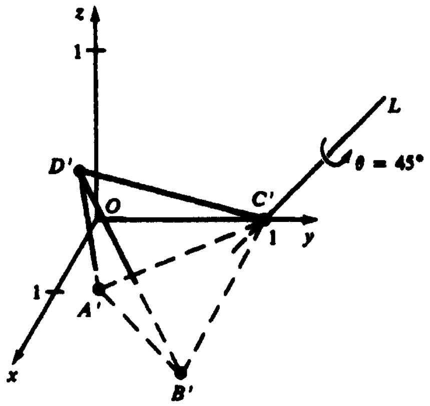


Fig. 6-7

6.5 Find a transformation $A_{\mathbf{V},\mathbf{N}}$ which aligns a vector **V** with a vector **N**.

SOLUTION

We form the transformation in two steps. First, align **V** with vector **K**, and second, align vector **K** with vector **N**. So from Prob. 6.2,

$$\mathbf{A}_{\mathbf{V},\mathbf{N}} = \mathbf{A}_{\mathbf{N}}^{-1} \cdot \mathbf{A}_{\mathbf{V}}$$

Referring to Prob. 6.12, we could also get $\mathbf{A}_{\mathbf{V},\mathbf{N}}$ by rotating **V** towards **N** about the axis $\mathbf{V} \times \mathbf{N}$.

6.6 Find the transformation for mirror reflection with respect to the xy plane.

SOLUTION

From Fig. 6-8, it is easy to see that the reflection of $P(x, y, z)$ is $P'(x, y, -z)$. The transformation that performs this reflection is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

6.7 Find the transformation for mirror reflection with respect to a given plane. Refer to Fig. 6-9.

SOLUTION

Let the plane of reflection be specified by a normal vector **N** and a reference point $P_0(x_0, y_0, z_0)$. To reduce the reflection to a mirror reflection with respect to the xy plane:

1. Translate P_0 to the origin:
2. Align the normal vector **N** with the vector **K** normal to the xy plane.
3. Perform the mirror reflection in the xy plane (Prob. 6.6).
4. Reverse steps 1 and 2.

So, with translation vector $\mathbf{V} = -x_0\mathbf{I} - y_0\mathbf{J} - z_0\mathbf{K}$

$$M_{\mathbf{N},P_0} = T_{\mathbf{V}}^{-1} \cdot A_{\mathbf{N}}^{-1} \cdot M \cdot A_{\mathbf{N}} \cdot T_{\mathbf{V}}$$

Here, $A_{\mathbf{N}}$ is the alignment matrix defined in Prob. 6.2. So if the vector $\mathbf{N} = n_1\mathbf{I} + n_2\mathbf{J} + n_3\mathbf{K}$, then from Prob.

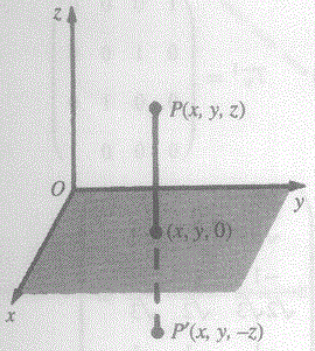


Fig. 6-8

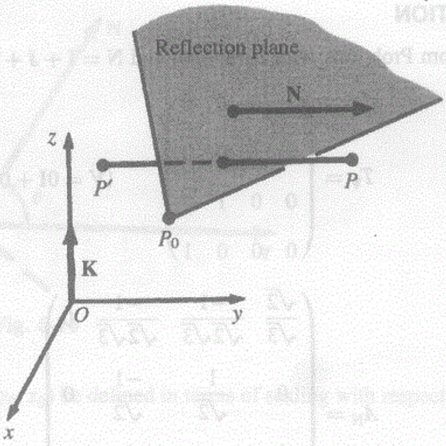


Fig. 6-9

6.2, with $|\mathbf{N}| = \sqrt{n_1^2 + n_2^2 + n_3^2}$ and $\lambda = \sqrt{n_2^2 + n_3^2}$, we find

$$A_{\mathbf{N}} = \begin{pmatrix} \frac{\lambda}{|\mathbf{N}|} & \frac{-n_1 n_2}{\lambda|\mathbf{N}|} & \frac{-n_1 n_3}{\lambda|\mathbf{N}|} & 0 \\ 0 & \frac{n_3}{\lambda} & \frac{-n_2}{\lambda} & 0 \\ \frac{n_1}{|\mathbf{N}|} & \frac{n_2}{|\mathbf{N}|} & \frac{n_3}{|\mathbf{N}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{\mathbf{N}}^{-1} = \begin{pmatrix} \frac{\lambda}{|\mathbf{N}|} & 0 & \frac{n_1}{|\mathbf{N}|} & 0 \\ \frac{-n_1 n_2}{\lambda|\mathbf{N}|} & \frac{n_3}{\lambda} & \frac{n_2}{|\mathbf{N}|} & 0 \\ \frac{-n_1 n_3}{\lambda|\mathbf{N}|} & \frac{-n_2}{\lambda} & \frac{n_3}{|\mathbf{N}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In addition

$$T_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally, from Prob. 6.6, the homogeneous form of M is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6.8 Find the matrix for mirror reflection with respect to the plane passing through the origin and having a normal vector whose direction is $\mathbf{N} = \mathbf{I} + \mathbf{J} + \mathbf{K}$.

SOLUTION

From Prob. 6.7, with $P_0(0, 0, 0)$ and $\mathbf{N} = \mathbf{I} + \mathbf{J} + \mathbf{K}$, we find $|\mathbf{N}| = \sqrt{3}$ and $\lambda = \sqrt{2}$. Then

$$T_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\mathbf{V} = 0\mathbf{I} + 0\mathbf{J} + 0\mathbf{K}) \qquad T_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{\mathbf{N}} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{\mathbf{N}}^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The reflection matrix is

$$M_{\mathbf{N},O} = T_{\mathbf{V}}^{-1} \cdot A_{\mathbf{N}}^{-1} \cdot M \cdot A_{\mathbf{N}} \cdot T_{\mathbf{V}}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$